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Magnetic Correlations on Fractals

Amnon Aharony,[1] Yuval Gefen,[2] and Yacov Kantor[3]

The critical behavior of magnetic spin models on various fractal structures is reviewed, with emphasis on branching and nonbranching Koch curves and Sierpiński gaskets and carpets. The spin correlation function is shown to have unusual exponential decays, e.g., of the form $\exp[-(r/\xi)^x]$, and to crossover to other forms at larger distances r. The various fractals are related to existing models for the backbone of the infinite incipient cluster at the percolation threshold, and conclusions are drawn regarding the behavior of spin correlations on these models.

KEY WORDS: Spin models; percolation; fractals; renormalization group; magnetic correlations.

1. INTRODUCTION

There are two main motivations to study spin models on fractal structures. First, fractals are fully, explicitly described geometric shapes, which one may consider as "hybrids" between standard (integer dimensionality) shapes, such as hypercubic lattices. A study of spin models on these shapes may thus shed light on the dependence of critical phenomena on the dimensionality, and identify additional geometrical parameters on which the universal properties of these phenomena may depend.[(1)] Secondly, as emphasized in these proceedings, many real physical structures exhibit self-similar (fractal) properties. In particular, we consider here the infinite incipient cluster at the percolation threshold.[(2,3)] Various geometrical models have been proposed in recent years to imitate this cluster, and it is of great interest to understand the effects of these different geometries on the magnetic properties of dilute magnets at the percolation threshold. Two extreme models have been proposed, i.e., the family of Sierpiński gaskets[(4)] and the "links and nodes"

[1] Department of Physics and Astronomy, Tel Aviv University, 69978 Tel Aviv, Israel.
[2] Institute of Theoretical Physics, University of California, Santa Barbara, California 93106.
[3] Exxon Corporate Research Laboratories, Route 22E, Annandale, New Jersey 08801.

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model.[5] In some respects, the two are combined in the "links and blobs" model.[6-8] In the present paper we review the critical properties of magnetic spin models on these various geometries. In Section 2 we review a series of renormalization group calculations on various fractals.[9-12] Models for the infinite incipient cluster at the percolation threshold are reviewed in Section 3, where consequences for the spin correlations on these models are also drawn.

2. SPIN MODELS ON FRACTALS

2.1. Nonbranching Quasilinear Koch Curves

An example of a nonbranching quasilinear Koch curve is shown in Fig. 1a. At each iteration of the construction, a segment is divided into three smaller segments, and the one in the middle is replaced by two new segments. Thus, the number of new segments is $N = 4$, the length scale changes by a factor $b = 3$, and the fractal dimensionality D, defined by $b^D = N$, is

$$D = \ln N/\ln b = \ln 4/\ln 3 \simeq 1.262 \tag{1}$$

The procedure is repeated many times, until the length of the basic segment becomes equal to some microscopic length (to be used as our unit of length). At this stage one places magnetic spins on the vertices, and one

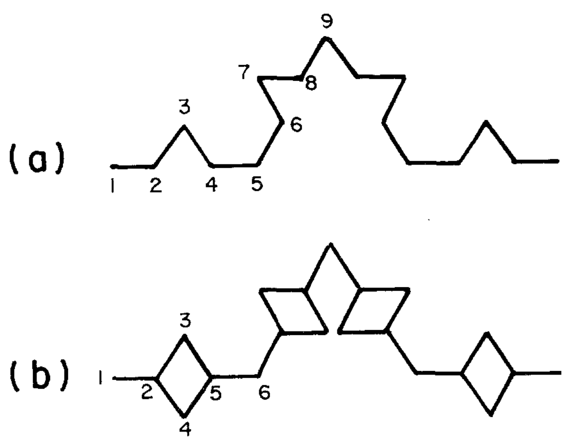

Fig. 1. Two examples of Koch curves: (a) nonbranching, (b) branching.

introduces an exchange interaction between nearest neighbor spins. For the Ising model, the Hamiltonian is

$$\mathscr{H} = -J \sum_i S_i S_{i+1} \tag{2}$$

where $S_i = \pm 1$, and where the index i counts the spins on the smallest microscopic scale.

We now perform a renormalization group (RG) transformation, by tracing over the "internal" spins on the smallest scale. In Fig. 1a, this implies tracing over the spins S_2, S_3, S_4, S_6, S_7, S_8, etc. For this example,

$$\operatorname*{Tr}_{S_2, S_3, S_4} \exp[K(S_1 S_2 + S_2 S_3 + S_3 S_4 + S_4 S_5)] \tag{3}$$

$$= \mathrm{Tr}(\cosh K)^4 \prod_{i=1}^{4} (1 + S_i S_{i+1} \tanh K) = 8(\cosh K)^4 [1 + S_1 S_5 (\tanh K)^4]$$

where $K = J/k_B T$, T being the temperature. This may be written $\exp(K' S_1 S_5 + C)$, with $\tanh K' = (\tanh K)^4$. For the general nonbranching Koch curve, the result is

$$\tanh K' = (\tanh K)^N \tag{4}$$

or $\ln \tanh K' = N \ln \tanh K = b^D \ln \tanh K$. Comparing this with the recursion relation for the correlation length,

$$\xi' = \xi / b \tag{5}$$

we thus identify

$$\xi = |\ln (\tanh K)|^{1/D} \simeq (2e^{-2K})^{-1/D} \tag{6}$$

Note that $|\ln \tanh K|^{-1} \simeq \frac{1}{2} e^{2K}$ is the correlation length for the one-dimensional Ising model.

If we repeat the transformation (4) l times, until the distance between the end points of the segment becomes $r = b^l$, then we find the spin–spin correlation function,

$$\langle S_0 S_r \rangle = \tanh L(l) = (\tanh K)^{N^l} = (\tanh K)^{r^D} \tag{7}$$

which may also be written

$$\langle S_0 S_r \rangle = \exp[-(r/\xi)^D] \tag{8}$$

If the distance was measured along the curved quasi-one-dimensional line, then Eq. (8) would simply reflect the usual exponential decay of Ising correlations in one dimension, $\exp(-r_I/\xi_I)$, with $r_I = r^D$, $\xi_I = |\ln(\tanh K)|^{-1}$. However, if experiments are done in the actual Euclidean d-dimensional space, in which the curve is embedded, then the correlation function is not simple exponential. In particular, the *structure factor*, $S(q)$, which is the Fourier transform of Eq. (8), *is not a Lorentzian*. For example, the Fourier transform of (8) with $D = 2$ behaves as $\xi^d \exp[-(q\xi/2)^2]$, and that with $D = 1$ as $\xi^d/[1 + (q\xi)^2]^{(d+1)/2}$.

The same general result applies for all discrete spin models. For example, one may consider the s-state Potts model,

$$\mathscr{H} = -J \sum_i \delta_{S_i S_{i+1}} \tag{9}$$

where $S_i = 1, 2, \ldots, s$, and find the recursion relation

$$X(K)' = X(K)^N \tag{10}$$

with

$$X(K) = (e^K - 1)/(e^K + s - 1) \tag{11}$$

These yield $\xi = |\ln |X(K)||^{-1/D}$, and everything else follows. One may also consider continuous spins. For the n-component model,

$$\mathscr{H} = -J \sum_i (\mathbf{S}_i \cdot \mathbf{S}_{i+1}) \tag{12}$$

where $\mathbf{S}_i$ is a unit vector of general direction, the low-temperature recursion relation is

$$\left(1 + \frac{a}{K'}\right) \simeq \left(1 + \frac{a}{K}\right)^N + O\left(\frac{1}{K^2}\right) \tag{13}$$

yielding $\xi \propto K^{1/D}$.

2.2. Branching Koch Curves

An example is shown in Fig. 1b. The relevant fractal dimensionality is

$$D = \ln N/\ln b = \ln 6/\ln 3 \simeq 1.631 \tag{14}$$

Again, we trace over the internal spins, to find

$$\exp(K' S_1 S_6 + C) = \operatorname*{Tr}_{S_2 S_3 S_4 S_5} \exp[K(S_1 S_2 + S_2 S_3 + S_2 S_4 + S_3 S_5 + S_4 S_5 + S_5 S_6)] \tag{15}$$

We first trace over S_3 and S_4, to find an effective coupling between S_2 and S_5. It is easy to see that this effective coupling is of the form $\exp(2\tilde{K}S_2S_5+\tilde{C})$, with $\tanh\tilde{K}=(\tanh K)^2$. The trace over S_2 and S_5 is now straightforward, and we find

$$\tanh K' = (\tanh K)^2 \tanh 2\tilde{K} \tag{16}$$

For low temperatures, $K \gg 1$, one has

$$\tanh\tilde{K} \simeq (1-2e^{-2\tilde{K}}) \simeq (1-2e^{-2K})^2 \simeq 1-4e^{-2K}$$

so that $e^{-2\tilde{K}} \simeq 2e^{-2K}$. Thus, $\tanh K' \simeq 1-2e^{-2K'} \simeq (1-2e^{-2K})^2 (1-8e^{-4K})$, i.e., $e^{-2K'} \simeq 2e^{-2K}+O(e^{-4K})$. Only the two "singly connected" bonds 1–2 and 5–6 contribute to $e^{-2K'}$ at order e^{-2K}. The "doubly connected" bonds contribute only at order e^{-4K}. The result may be easily generalized:

$$e^{-2K'} \simeq L_1 e^{-2K} + O(e^{-4K}) \tag{17}$$

where L_1 is the number of singly connected bonds in the basic iterated structure. It is now easy to obtain $\xi_1 = |\ln(\tanh K)|^{-1/x_1} \simeq (2e^{-2K})^{-1/x_1}$, with

$$x_1 = \ln L_1/\ln b \tag{18}$$

(i.e., $x_1 = \ln 2/\ln 3 \simeq 0.631$ in our example).

As before, the correlation function becomes

$$\langle S_0 S_r\rangle \simeq \exp[-(r/\xi_1)^{x_1}] \tag{19}$$

with corrections of order e^{-4K}.

It is interesting to keep track of the corrections of order e^{-4K}. In our example, $\tanh 2\tilde{K} \simeq 1-2e^{-4\tilde{K}} \simeq 1-8e^{-4K} \simeq (1-2e^{-4K})^4 \simeq (\tanh 2K)^4$, with corrections of order e^{-6K}. To this order, Eqs. (16) or (17) are generalized to

$$\tanh K' = (\tanh K)^{L_1}(\tanh 2K)^{L_2} + O(e^{-6K}) \tag{20}$$

where L_2 is the number of pairs of "doubly connected" bonds, i.e., bonds such that the connection between the end spins (1 and 6 in Fig. 1b) is broken if the two bonds in a pair are broken (but not if one of them is broken).

We can thus define a new length,

$$\xi_2 = |\ln(\tanh 2K)|^{-1/x_2} \simeq (2e^{-4K})^{-1/x_2} \tag{21}$$

with

$$x_2 = \ln L_2 / \ln b \tag{22}$$

and the correlation function becomes

$$\langle S_0 S_r \rangle = \exp\left[-\left(\frac{r}{\xi_1}\right)^{x_1} - \left(\frac{r}{\xi_2}\right)^{x_2} + O(e^{-6K}) \right] \tag{23}$$

The correlations will be dominated by the "singly connected" bonds only if $(r/\xi_1)^{x_1} \gg (r/\xi_2)^{x_2}$. Thus, if $x_2 > x_1$ then Eq. (19) applies only in the range $r < e^{2K/(x_2 - x_1)}$. For larger r, or for lower K (higher temperature), the second term in Eq. (23) wins.

In the example of Fig. 1b, $x_2 = \ln 4/\ln 3 = 2x_1$, and the crossover discussed here is expected at $r \sim e^{2K/x_1} \sim \xi_1$. It is easy to construct examples with $L_1 = 0$ (i.e., $x_1 = -\infty$), when the second term always wins, or ones with $x_2 > x_1$, when Eq. (19) is always valid (as in the case for the nonbraching curves). One could also have $L_1 = L_2 = 0$, so that the correlations are determined by higher-order terms.

The same phenomena are predicted for all discrete models, e.g., the s-state Potts model. However, the situation changes for the continuous spin models: Here, Eq. (16) is replaced by

$$\left(1 - \frac{a}{K'}\right) = \left(1 - \frac{a}{K}\right)^2 \left(1 - \frac{a}{2\tilde{K}}\right) + O\left(\frac{1}{K^2}\right) \tag{24}$$

with $(1 - a/\tilde{K}) = (1 - a/K)^2 + O(1/K^2)$. Thus, $1/\tilde{K} \simeq 2/K$, and the factor involving $\tilde{K}$ in Eq. (24) contributes to the *leading* order in $1/K$ (and not only to the *next* order, as in the discrete cases). In fact, it is easy to see that the $(1/K)$'s add like resistors (in series or in parallel), and that

$$1/K' = R/K + O(1/K^2) \tag{25}$$

where R is the resistance of the basic iterated structure (equal to 3 for Fig. 1b). Defining $\xi \propto K^\nu$, we thus identify

$$1/\nu = \tilde{\zeta} = \ln R / \ln b$$

and

$$\langle \mathbf{S}_0 \cdot \mathbf{S}_r \rangle \simeq \exp[-(r/\xi)^{\tilde{\zeta}}] \tag{26}$$

for continuous spins.

2.3. Sierpiński Gaskets

Several iterations of the Sierpiński gasket in two dimensions are shown in Fig. 2. The gasket is intermediate between the quasi-one-dimensional curves and those in higher dimensions, as it is finitely ramified but its minimum order of ramification is larger than 2.[4,10] In d dimensions, the fractal dimensionality of the gasket is

$$D = \ln(d+1)/\ln 2 \tag{27}$$

and its minimum order of ramification is $R_{\min} = d + 1$.

As in the previous cases, we solve the Ising model on the gasket by tracing over the internal spins. For the two-dimensional example of Fig. 2, this amounts to tracing over three spins. The resulting recursion relation is[1,10]

$$e^{4K'} = (e^{8K} - e^{4K} + 4)/(e^{4K} + 3) \tag{28}$$

and for small $t = e^{-4K}$ its solution is $t(l) = [4(l_0 - l)]^{-1} + O[(l_0 - l)^{-2}]$, where $4l_0 = 1/t(0) = e^{4K}$. Iterating until the effective correlation length becomes $\xi(l) = \xi/2^l = O(1)$, when we also expect that $t(l) = O(1)$, i.e., $l \simeq l_0$, we find

$$\xi \propto \exp[\tfrac{1}{4} \ln 2 \exp(4K)] \tag{29}$$

We solved various discrete spin models on gaskets at $d = 2$ and $d = 3$,[10] and always found expressions like (29), in which ξ is exponential in e^{2K} (rather than a power of e^{2K}, as in the quasi-one-dimensional cases). Note that (in the spirit of Section 2.2) the gasket contains neither "singly connected" bonds, nor "doubly connected" bonds, etc. It represents the extreme case of loops within loops ad infinitum.

The correlation function between two spins, at a distance r, which occupy the corners of a large triangle, may be estimated as $\tanh K(l)$, where $r = 2^l$. For $r \ll \xi$, i.e., $l \ll l_0$, $\tanh K(l) \simeq 1 - 2e^{-2K(l)} \simeq 1 - (l_0 - l)^{-1/2} \simeq 1 - [\ln(\xi/r)/\ln 2]^{-1/2}$ or

$$\langle S_0 S_r \rangle \simeq \exp\{-[\ln 2/\ln(\xi/r)]^{1/2}\} \tag{30}$$

Again, this is a very unusual form.

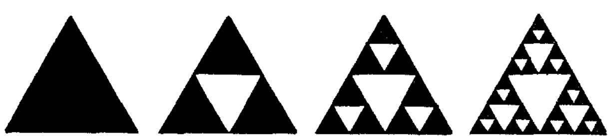

Fig. 2. The Sierpiński gasket, $d = 2$.

The behavior of continuous spin models is again determined by that of a resistor network, and we find $\xi \sim T^{-1/\tilde{\zeta}}$, with[4,10]

$$\tilde{\zeta} = \ln[(d+3)/(d+1)]/\ln 2 \tag{31}$$

2.4. Sierpiński Carpets

All the systems reviewed so far have a finite order of ramification, and thus end up having no phase transition at any finite temperature. We have also studied a series of many Sierpiński carpets,[1,11,12] and found a finite transition temperature for the discrete spin models whenever the order of ramification was infinite. In order to characterize the geometry of the carpets one also needs the connectivity Q and the lacunarity L.[1,2,11,12] An approximate Migdal–Kadanoff RG scheme yields exponents which depend on all these geometrical factors (and not only on D). One must therefore generalize one's characterization of universality classes, to include D, Q, L, and probably other factors. Note that the critical properties of the branching Koch curves were also not solely characterized by D (and one needed to add the exponents x_1, x_2, etc.).

Addressing the question of the relation between fractals and the analytic continuation of hypercubic lattices we found that the exponents of the two become the same (at least in $d = 1 + \varepsilon$ dimensions) when the lacunarity of the fractals approaches zero, which is as close to being translationally invariant as possible.[12]

Concluding this section, we may say that the general question of magnetic correlations on fractals has only begun to be studied, and many questions remain open for the future.

3. MODELS FOR THE BACKBONE OF THE INFINITE CLUSTER

The propagation of correlations (as well as the flow of a current) through the infinite cluster involves only its *backbone*, and not the dangling ("dead end") bonds. There exist many numerical and experimental studies which show that the backbone is self-similar, with the fractal dimensionality

$$D_B = d - \beta_B/\nu_p \tag{32}$$

where $P_B \propto (p - p_c)^{\beta_B}$ is the probability to belong to the backbone, and $\xi_p \propto (p - p_c)^{-\nu_p}$ is the pair connectedness length.[14,15] Moreover, its order of ramification is finite, but it is not quasi-one-dimensional.[15] The Sierpiński gaskets are the simplest nonrandom fractals which are compatible with these properties. Moreover, their fractal dimensionalities, Eq. (27), are very close to those of the backbones, for $1 \leqslant d \leqslant 4$. The gaskets were therefore proposed as models for the backbone.[4]

An explicit calculation of the resistivity of the gaskets, related to Eq. (31), also gave reasonable estimates for the appropriate exponents. However, these estimates are very rough, and are excluded by recent accurate simulations.

As mentioned above, the gaskets have the extreme property of not having any singly connected, doubly connected, etc., bonds. This led to the prediction (29), which seems to indicate too strong a divergence of ξ.

The other extreme model, by Skal and Shklovskii,[5] describes the backbone as a superlattice made by nodes separated by a distance of order ξ_p, connected by curved quasi-one-dimensional links, built of $L \sim (p - p_c)^{-\zeta}$ bonds (Fig. 3a). On length scales $r < \xi_p$, the correlation between two spins is only via these quasi-one-dimensional links. On these length scales, all the "mass" of the backbone is on the links, hence $L \sim \xi_p^{D_B}$ and $\zeta = D_B \nu_p$.

Skal and Shklovskii assumed that the links are nonbranching. For $r < \xi_p$, this yields our Eqs. (6) and (8). However, there exist exact proofs[7,8] that at p_c one should have $\xi_1 \propto (e^{2K})^{\nu_p}$, in contrast to Eq. (6) (since $D_B \neq 1/\nu_p$). Moreover, Eqs. (6) and (13) predict the same exponents for discrete and continuous spin models, and this does not agree with experiments.[8]

The "links and nodes" model was therefore modified into the "links and blobs" one[6-8] (Fig. 1b). The "blobs" contain "multiply connected" bonds, and they are introduced in a self-similar way.[8] Within this model, the correlation between two spins at distance $r < \xi_p$ is still via a single link, but this link is now a *branching* curve. Therefore, our Section 2.2 may be used. As Coniglio[7,8] noted, we now have at our disposal the three exponents D_B, x_1, and $\tilde{\zeta}$, and therefore we can fit the fractal dimensionality and the behavior of discrete and continuous spin models.

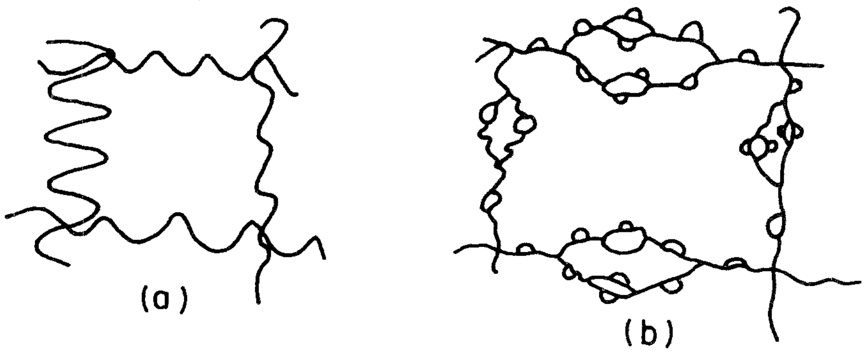

Fig. 3. Models for the backbone: (a) "links and nodes," (b) "links and blobs."

Combining Eq. (18) with Coniglio's result $\xi \sim (e^{2K})^{\nu_p}$ we now identify

$$x_1 = 1/\nu_p \tag{33}$$

Coniglio's proof may also be generalized[16,13] to yield $x_2 = 2x_1$. Thus, the links and blobs model predicts Eq. (23), with a crossover from $\exp[-(r/\xi_1)^{1/\nu_p}]$ to $\exp[-(r/\xi_2)^{2/\nu_p}] \sim \exp[-(r/\xi_1)^{2/\nu_p}]$ at $r \sim e^{2K\nu_p} \sim \xi_1$. For larger values of r it is reasonable to expect higher-order terms, e.g., $(r/\xi_1)^{3/\nu_p}$, to become equally important. These will probably cut the correlation function off rather quickly beyond $r > \xi_1$.

So far we discussed only the magnetic correlation function between two spins on the infinite cluster, at p_c. In fact, the same correlation function is expected for any two spins which belong to the same percolating cluster. The probability that the two sites (at the origin and at a distance r) both belong to the cluster is given by the percolation correlation function, $G(r)$. At p_c, this function has the form

$$G(r) \sim 1/r^{d-2+\eta_p} \tag{34}$$

Thus, we conclude that at p_c the average spin correlation function is

$$\overline{\langle S_0 S_r \rangle} \propto \frac{1}{r^{d-2+\eta_p}} \exp\left[-\left(\frac{r}{\xi_1}\right)^{1/\nu_p}\right] \tag{35}$$

for $r < \xi_1$, with a faster decay for $r > \xi_1$.

The situation is further complicated for $p \neq p_c$, when ξ_p is finite. We still expect Eq. (35) to hold for $r < \xi_p$. The function (35) will hold for all $r < \xi_p$ if $\xi_p < \xi_1$. A modified behavior is expected for $\xi_1 < r < \xi_p$ if $\xi_1 < \xi_p$. The line $\xi_1 \sim \xi_p$, or $e^{2K} \sim (p - p_c)^{-1}$, indeed represents a crossover, as predicted before.[7,8] However, the behavior we expect on the two sides of this line has several new features.

Experimentally, the spin correlation function is masured via its Fourier transform, the structure factor $S(q)$.[17] It is far from trivial to perform such a transform on Eq. (35). In any case, it is clear that for $q\xi_1 \ll 1$ one may write

$$S(q) \propto [1 + C(q\xi_1)^2 + O((q\xi_1)^4)]^{-1} \tag{36}$$

while for $q\xi_1 \gg 1$ one recovers the percolation result

$$S(q) \propto 1/q^{2-\eta_p} \tag{37}$$

In general, $S(q)$ is not a Lorentzian. A detailed discussion of $S(q)$ will be given elsewhere.[13]

In conclusion, we have shown that the "links and blobs" model, as any other model based on self-similar branching Koch curves, yields Eq. (35). It would be very interesting to test details of this correlation function, both numerically and experimentally. It should be emphasized that it is not clear if this model fully describes the backbone. If, instead, there are loops within loops on all scales, as in the gaskets, then the picture is drastically modified. The true description, which is probably intermediate between the two pictures, remains to be explored in the future.

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